

# A lattice Poisson algebra for the Pohlmeyer reduction of the $AdS_5 \times S^5$ superstring

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**Abstract.** The Poisson algebra of the Lax matrix associated with the Pohlmeyer reduction of the  $AdS_5 \times S^5$  superstring is computed from first principles. The resulting non-ultralocality is mild, which enables to write down a corresponding lattice Poisson algebra.

## 1 Introduction

We recently showed in [1] that the Poisson algebra of the Lax matrix associated with symmetric space sine-Gordon models, defined through a gauged Wess-Zumino-Witten action with an integrable potential [2], admits an integrable lattice discretization. In the present letter we compute the  $r/s$ -matrix structure [3] associated with the Pohlmeyer reduction of  $AdS_5 \times S^5$  superstring theory [4, 5] directly from its representation in terms of a fermionic extension of a gauged WZW action with an integrable potential. We similarly find that it is precisely of the type which, after regularization as in [6], admits an integrable lattice discretization of the general form identified in [7, 8].

## 2 Canonical analysis and Hamiltonian

To begin with we briefly recall some usual notations. We refer the reader to [4] for more details concerning this setup. The superalgebra  $\mathfrak{f} = \mathfrak{psu}(2, 2|4)$  admits a  $\mathbb{Z}_4$ -grading,  $\mathfrak{f} = \mathfrak{f}^{(0)} \oplus \mathfrak{f}^{(1)} \oplus \mathfrak{f}^{(2)} \oplus \mathfrak{f}^{(3)}$  where  $\mathfrak{g} = \mathfrak{f}^{(0)} = \mathfrak{so}(4, 1) \oplus \mathfrak{so}(5)$ . Let  $G$  denote the corresponding Lie group. The supertrace is compatible with the  $\mathbb{Z}_4$ -grading, in the sense that  $\text{Str}(A^{(m)}B^{(n)}) = 0$  for  $m + n \neq 0 \pmod{4}$ . The

reduced theory relies on the element  $T = \frac{i}{2} \text{diag}(1, 1, -1, -1, 1, 1, -1, -1) \in \mathfrak{f}^{(2)}$ . It defines a  $\mathbb{Z}_2$ -grading of  $\mathfrak{f}$  with  $\mathfrak{f}^{[0]} = \text{Ker}(\text{Ad}_T)$  and  $\mathfrak{f}^{[1]} = \text{Im}(\text{Ad}_T)$ . Elements of  $\mathfrak{f}^{[0]}$  commute with  $T$  while those of  $\mathfrak{f}^{[1]}$  anti-commute with  $T$  and we have  $\text{Str}(A^{[0]}B^{[1]}) = 0$ . Finally, projectors on  $\mathfrak{f}^{[0]}$  and  $\mathfrak{f}^{[1]}$  are given respectively by  $P^{[0]} = -[T, [T, \cdot]_+]_+$  and  $P^{[1]} = -[T, [T, \cdot]]$ . Let  $\mathfrak{h} = \mathfrak{g}^{[0]}$  be the subalgebra in  $\mathfrak{g}$  of elements commuting with  $T$ . The corresponding Lie group  $H$  is  $[SU(2)]^4$ .

Our starting point is the field theory introduced in [4]. It corresponds to a fermionic extension of a  $G/H$  gauged WZW with a potential term. The action we start with is, taking  $\epsilon^{\tau\sigma\xi} = 1$ ,

$$\begin{aligned} \mathcal{S} = & \frac{1}{2} \int d\tau d\sigma \text{Str}(g^{-1} \partial_+ g g^{-1} \partial_- g) + \frac{1}{3} \int d\tau d\sigma d\xi \epsilon^{\alpha\beta\gamma} \text{Str}(g^{-1} \partial_\alpha g g^{-1} \partial_\beta g g^{-1} \partial_\gamma g) \\ & - \int d\tau d\sigma \text{Str}(A_+ \partial_- g g^{-1} - A_- g^{-1} \partial_+ g + g^{-1} A_+ g A_- - A_+ A_-) \\ & + \frac{1}{2} \int d\tau d\sigma \text{Str}(\psi_L [T, D_+ \psi_L] + \psi_R [T, D_- \psi_R]) \\ & + \int d\tau d\sigma (\mu^2 \text{Str}(g^{-1} T g T) + \mu \text{Str}(g^{-1} \psi_L g \psi_R)). \end{aligned}$$

The fields  $g$ ,  $\psi_R$ ,  $\psi_L$  and the gauge fields  $A_\pm$  respectively take values in  $G$ ,  $\mathfrak{f}^{(1)[1]}$ ,  $\mathfrak{f}^{(3)[1]}$  and in  $\mathfrak{h}$ . The covariant derivatives are  $D_\pm = \partial_\pm - [A_\pm, \cdot]$  with  $\partial_\pm = \partial_\tau \pm \partial_\sigma$ .

Generalizing the analysis of [9] to the case considered here, one finds that the phase space is spanned by the fields  $(g, \mathcal{J}_L, A_\pm, P_\pm, \psi_L, \psi_R)$ . The field  $\mathcal{J}_L$  corresponds to the left-invariant WZW current. Alternatively, one can use instead the right-invariant current  $\mathcal{J}_R$ , related to  $\mathcal{J}_L$  by

$$\mathcal{J}_R = -2\partial_\sigma g g^{-1} + g \mathcal{J}_L g^{-1}.$$

The fields  $P_\pm$  are the canonical momenta of  $A_\pm$ . The non-vanishing Poisson brackets are

$$\begin{aligned} \{\mathcal{J}_{L\mathbf{1}}(\sigma), \mathcal{J}_{L\mathbf{2}}(\sigma')\} &= [C_{\mathbf{12}}^{(00)}, \mathcal{J}_{L\mathbf{2}}] \delta_{\sigma\sigma'} + 2C_{\mathbf{12}}^{(00)} \partial_\sigma \delta_{\sigma\sigma'}, \\ \{\mathcal{J}_{R\mathbf{1}}(\sigma), \mathcal{J}_{R\mathbf{2}}(\sigma')\} &= -[C_{\mathbf{12}}^{(00)}, \mathcal{J}_{R\mathbf{2}}] \delta_{\sigma\sigma'} - 2C_{\mathbf{12}}^{(00)} \partial_\sigma \delta_{\sigma\sigma'}, \\ \{\mathcal{J}_{L\mathbf{1}}(\sigma), g_{\mathbf{2}}(\sigma')\} &= -g_{\mathbf{2}} C_{\mathbf{12}}^{(00)} \delta_{\sigma\sigma'} \\ \{\mathcal{J}_{R\mathbf{1}}(\sigma), g_{\mathbf{2}}(\sigma')\} &= -C_{\mathbf{12}}^{(00)} g_{\mathbf{2}} \delta_{\sigma\sigma'} \\ \{A_{\pm\mathbf{1}}(\sigma), P_{\pm\mathbf{2}}(\sigma')\} &= C_{\mathbf{12}}^{(00)[00]} \delta_{\sigma\sigma'}, \\ \{\psi_{R\mathbf{1}}(\sigma), \psi_{R\mathbf{2}}(\sigma')\} &= [T_{\mathbf{2}}, C_{\mathbf{12}}^{(13)}] \delta_{\sigma\sigma'}, \\ \{\psi_{L\mathbf{1}}(\sigma), \psi_{L\mathbf{2}}(\sigma')\} &= [T_{\mathbf{2}}, C_{\mathbf{12}}^{(31)}] \delta_{\sigma\sigma'}. \end{aligned}$$

In these expressions  $C_{\mathbf{12}}^{(ij)} \in \mathfrak{f}^{(i)} \otimes \mathfrak{f}^{(j)}$  are the components of the tensor Casimir (see [10] for its properties) in the decomposition  $C_{\mathbf{12}} = C_{\mathbf{12}}^{(00)} + C_{\mathbf{12}}^{(13)} + C_{\mathbf{12}}^{(22)} + C_{\mathbf{12}}^{(31)}$  with respect to the  $\mathbb{Z}_4$ -grading. The component  $C_{\mathbf{12}}^{(00)[00]}$  is defined in a similar way relative to the  $\mathbb{Z}_2$ -grading.

The standard analysis shows that there is a total of four constraints,

$$\chi_1 = P_+, \quad \chi_2 = P_-, \quad (2.2a)$$

$$\chi_3 = \mathcal{J}_R^{[0]} + A_+ - A_- - \frac{1}{2}[\psi_L, [T, \psi_L]], \quad \chi_4 = \mathcal{J}_L^{[0]} + A_+ - A_- + \frac{1}{2}[\psi_R, [T, \psi_R]]. \quad (2.2b)$$

The extended Hamiltonian, which has weakly vanishing Poisson brackets with the constraints (2.2), is

$$\begin{aligned} H = \int d\sigma & \left( \frac{1}{4} \text{Str}(\mathcal{J}_L^2 + \mathcal{J}_R^2) + \text{Str}(\mathcal{J}_R^{[0]} A_+ - \mathcal{J}_L^{[0]} A_-) + \frac{1}{2} \text{Str}[(A_+ - A_-)^2] \right. \\ & - \frac{1}{2} \text{Str}(\psi_L [T, \partial_\sigma \psi_L - [A_+, \psi_L]]) - \frac{1}{2} \text{Str}(\psi_R [T, -\partial_\sigma \psi_R - [A_-, \psi_R]]) \\ & \left. - \mu^2 \text{Str}(g^{-1} T g T) - \mu \text{Str}(g^{-1} \psi_L g \psi_R) + v_+ P_+ + v_- P_- + \lambda(\chi_3 - \chi_4) \right) \end{aligned} \quad (2.3)$$

with  $v_+ - v_- = \partial_\sigma(A_+ + A_-) - [A_+, A_-]$ . The combination  $\chi_3 - \chi_4$  of the constraints generates a gauge invariance.

### 3 Continuum and lattice Poisson algebras

Up to a gauge transformation, the equations of motion for the fields  $(\mathcal{J}_L, g, \psi_L, \psi_R)$  under the Hamiltonian (2.3) are equivalent to the zero curvature equation  $\{\mathcal{L}, H\} = \partial_\sigma \mathcal{M} + [\mathcal{M}, \mathcal{L}]$  for the following Lax connection [4]

$$\mathcal{L}(z) = -\frac{1}{2}\mathcal{J}_L - \frac{1}{2}z\sqrt{\mu}\psi_R - \frac{1}{2}z^2\mu T + \frac{1}{2}z^{-1}\sqrt{\mu}g^{-1}\psi_L g + \frac{1}{2}z^{-2}\mu g^{-1}Tg, \quad (3.1a)$$

$$\mathcal{M}(z) = -\frac{1}{2}\mathcal{J}_L + A_- - \frac{1}{2}z\sqrt{\mu}\psi_R - \frac{1}{2}z^2\mu T - \frac{1}{2}z^{-1}\sqrt{\mu}g^{-1}\psi_L g - \frac{1}{2}z^{-2}\mu g^{-1}Tg. \quad (3.1b)$$

The field  $A_+$  entering the equations appears as an arbitrary element of  $\mathfrak{h}$ . We now have all the ingredients needed to compute the Poisson bracket of the Lax matrix (3.1a). The result reads

$$\begin{aligned} 4\{\mathcal{L}_1(z_1), \mathcal{L}_2(z_2)\} &= [r_{12}(z_1, z_2), \mathcal{L}_1(z_1) + \mathcal{L}_2(z_2)]\delta_{\sigma\sigma'} \\ &+ [s_{12}(z_1, z_2), \mathcal{L}_1(z_1) - \mathcal{L}_2(z_2)]\delta_{\sigma\sigma'} + 2s_{12}(z_1, z_2)\partial_\sigma\delta_{\sigma\sigma'}, \end{aligned} \quad (3.2)$$

where the kernels of the  $r/s$ -matrices are given by

$$r_{12}(z_1, z_2) = \frac{z_2^4 + z_1^4}{z_2^4 - z_1^4} C_{12}^{(00)} + \frac{2z_1 z_2^3}{z_2^4 - z_1^4} C_{12}^{(13)} + \frac{2z_1^2 z_2^2}{z_2^4 - z_1^4} C_{12}^{(22)} + \frac{2z_1^3 z_2}{z_2^4 - z_1^4} C_{12}^{(31)}, \quad (3.3a)$$

$$s_{12}(z_1, z_2) = C_{12}^{(00)}. \quad (3.3b)$$

One can check explicitly that the kernels (3.3) coincide exactly with the ones that would be obtained from the generalization of the alleviation procedure proposed in [1] to semi-symmetric space  $\sigma$ -models. This is simply a matter of replacing the twisted inner product on the twisted loop

algebra considered in [11] by the trigonometric one and to compute the corresponding kernels as explained in [1].

An important property of the above  $r/s$ -matrix structure is that  $s$  is simply the projection onto the subalgebra  $\mathfrak{g}$ . In this case, the corresponding Poisson algebra (3.2) can be discretized following [6] by introducing a skew-symmetric solution  $\alpha \in \text{End } \mathfrak{g}$  of the modified classical Yang-Baxter equation on  $\mathfrak{g}$ . Then the matrices

$$a_{\underline{1}\underline{2}} = (r + \alpha)_{\underline{1}\underline{2}}, \quad b_{\underline{1}\underline{2}} = (-s - \alpha)_{\underline{1}\underline{2}}, \quad c_{\underline{1}\underline{2}} = (-s + \alpha)_{\underline{1}\underline{2}}, \quad d_{\underline{1}\underline{2}} = (r - \alpha)_{\underline{1}\underline{2}},$$

satisfy all the requirements of [7, 8] in order to define the following consistent lattice algebra,

$$4\{\mathcal{L}_{\underline{1}}^n, \mathcal{L}_{\underline{2}}^m\} = a_{\underline{1}\underline{2}}\mathcal{L}_{\underline{1}}^n\mathcal{L}_{\underline{2}}^m\delta_{mn} - \mathcal{L}_{\underline{1}}^n\mathcal{L}_{\underline{2}}^m d_{\underline{1}\underline{2}}\delta_{mn} + \mathcal{L}_{\underline{1}}^n b_{\underline{1}\underline{2}}\mathcal{L}_{\underline{2}}^m\delta_{m+1,n} - \mathcal{L}_{\underline{2}}^m c_{\underline{1}\underline{2}}\mathcal{L}_{\underline{1}}^n\delta_{m,n+1}.$$

This algebra reduces to (3.2) in the continuum limit (see [1]). The corresponding algebra for the monodromy may be found in [1].

## 4 Conclusion

We have constructed a quadratic lattice Poisson algebra associated with the fermionic extension of the  $(SO(4, 1) \times SO(5))/[SU(2)]^4$  gauged WZW model with an integrable potential. The fact that one is able to write down such a lattice algebra is quite appealing and in sharp contrast with what happens for the canonical Poisson structure of the  $AdS_5 \times S^5$  superstring [10]. Indeed, it brings hope of being able to construct a lattice quantum algebra related to the Pohlmeyer reduction of the  $AdS_5 \times S^5$  superstring. The precise link of this Pohlmeyer reduction with the alleviation procedure presented in [1] is under study.

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